JOURNAL OF MULTIVARIATE ANALYSIS 13, 366–374 (1983)

Weak Convergence of Linear Forms in $D[0, 1]$

PETER Z. DAFFER AND ROBERT L. TAYLOR*

Vanderbilt University and University of South Carolina

Communicated by M. M. Rao

Convergence in probability of the linear forms $\sum_{k=1}^{\infty} a_{nk} X_k$ is obtained in the space $D[0, 1]$, where (X_k) are random elements in $D[0, 1]$ and (a_{nk}) is an array of real numbers. These results are obtained under varying hypotheses of boundedness conditions on the moments and conditions on the mean oscillation of the random elements (X_n) on subintervals of a partition of $[0, 1]$. Since the hypotheses are in general much less restrictive than tightness (or convex tightness), these results represent significant improvements over existing weak laws of large numbers and convergence results for weighted sums of random elements in $D[0, 1]$. Finally, comparisons to classical hypotheses for Banach space and real-valued results are included.

1. INTRODUCTION AND PRELIMINARIES

Using uncorrelation conditions and conditions on the mean oscillation of X_n on subintervals of a partition of $[0, 1]$, weak convergence results are obtained for weighted sums of random elements (X_n) in $D[0, 1]$ which are significant improvements on Theorem 3 and 5 of Daffer and Taylor [3], Theorem 1 of Taylor and Daffer [6] and Theorem 3.1 of Daffer [2]. Tightness of the sequence (X_n) is in general not implied by these hypotheses, and hence the results are different from the traditional results for $D[0, 1]$ where convex tightness (a strengthening of the notion of tightness in which the compact sets involved are also required to be convex) is used in a crucial manner.

The properties of the space $D = D[0, 1]$ are well known and are treated at length in Billingsley [1] and Parthasarathy [4]. Here, D will always be given the Skorohod topology, with respect to which the supremum norm $\|x\| =$

Received May 11, 1981; revised March 1, 1982.

AMS 1980 subject classification numbers: Primary 60B12; Secondary 60F15.

Key words and phrases: Random elements in $D[0, 1]$, linear forms, convergence in probability, weighted sums, laws of large numbers, Toeplitz array, and integral conditions.

*Research by this author was supported in part by the Air Force Office of Scientific Research under Contract F49620-79-C-0140 and Grant AFOSR-81-0166.

0047-259X/83 $3.00

$\sup_{0\leqslant t\leqslant 1}|x(t)|$, $x \in D$, is known to be a Borel measurable function. A random element in D is a measurable map X from a probability space $(\Omega, \mathscr{A}, P)$ into D. The mathematical expectation EX of a random element X is defined pointwise by $(EX)(t) = E[X(t)], 0 \leqslant t \leqslant 1$. A sufficient condition for $EX \in D$ is that $E\|X\| < \infty$.

A partition π_m of $[0, 1]$ will always refer to a finite partition $0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = 1$, $m \in N$ (N stands for the natural numbers). The subintervals of the partition π_m will be denoted by $I_i = [t_{i-1}, t_i)$, $i = 1,..., m-1$; $I_m = [t_{m-1}, 1]$. The *norm* $\|\pi_m\|$ of a partition π_m is $\max\{t_i - t_{i-1}: i = 1,..., m\}$. For a set A, I_A denotes the indicator function of A.

A Toeplitz array $\{a_{nk}\}$, n, $k \in N$, is a double sequence of real numbers satisfying

(i) $\sum_{k=1}^{\infty} |a_{nk}| \leqslant 1$, $\forall\, n \in N$, and

(ii) $\lim_{n\to\infty} a_{nk} = 0$, $\forall\, k \in N$.

Let (X_n) be a sequence of random elements in D. For future reference in obtaining the main results some conditions are listed which the sequence (X_n), $n \in N$, can be required to satisfy.

1.1. Condition (T) is satisfied if, to every $\varepsilon > 0$, there is a compact $K \subset D$ such that $\sup_n E\|X_n I_{[X_n \notin K]}\| \leqslant \varepsilon$.

1.2. Condition (MT) is satisfied if, to every $\varepsilon > 0$, there is a partition π_m such that $\sup_n E[\max_{i=1,...,m} \sup_{t\in I_i} |X_n(t) - X_n(t_{i-1})|] \leqslant \varepsilon$.

1.3. Condition (mt) is satisfied if, to every $\varepsilon > 0$, there is a partition π_m such that $\sup_n \max_{i=1,...,m} E[\sup_{t\in I_i} |X_n(t) - X_n(t_{i-1})|] \leqslant \varepsilon$.

1.4. The sequence (X_n) is said to be *pointwise uncorrelated* if $E[X_n^2(t)] < \infty$, $n \in N$, $0 \leqslant t \leqslant 1$, and $\mathrm{Cov}(X_n(t), X_k(t)) = E[(X_n(t) - EX_n(t))(X_k(t) - EX_k(t))] = 0$, when $n \neq k$, for each $0 \leqslant t \leqslant 1$.

1.5. Condition (uc) is satisfied if for any s, t, $0 \leqslant s < t \leqslant 1$, $(\sup_{s\leqslant u<t} |X_n(u) - X_n(s)|)_{n\in N}$ is a sequence of uncorrelated random variables, for each $i = 1,..., m$.

1.6. Condition (AU) is satisfied if for any s, t, $0 \leqslant s < t \leqslant 1$, $(|X_n(t-0) - X_n(s)|)_{n\in N}$ is a sequence of uncorrelated random variables.

1.7. Condition (C) is satisfied if, to every $\varepsilon > 0$, there is a partition π_m such that $E[\max_{i=1,...,m} \sup_{t\in I_i} |X_n(t)\, X_k(t) - X_n(t_{i-1})\, X_k(t_{i-1})|] \leqslant \varepsilon$, whenever $n \neq k$, n, $k \in N$.

In Sections 2 and 3 relationships among the conditions are discussed, and the different uncorrelation concepts (including Banach space versions) are contrasted.

2. Weak Convergence of Weighted Sums of Random Elements in D

The weak convergence theorems in this section are of two types: Theorems 2.1, 2.2 and 2.3 provide conditions which are sufficient for convergence in probability of the weighted sums at each point t of $[0, 1]$ to imply uniform convergence in probability on $[0, 1]$. Theorem 2.4 provides conditions which directly imply convergence in probability of the weighted sums.

THEOREM 2.1. *Let (X_n) be a sequence of random elements in D satisfying (mt) and(uc). Let $\{a_{nk}\}$, $k = 1,..., k_n$; $n \in N$, be an array of real numbers satisfying $\sum_{k=1}^{k_n} |a_{nk}| \leqslant C < \infty$, for each n, and $\lim_{n\to\infty} \sum_{k=1}^{k_n} a_{nk}^2 E(\|X_k\|^2) = 0$. Then,*

$$\lim_{n\to\infty} \left\| \sum_{k=1}^{k_n} a_{nk} X_k \right\| = 0, \qquad \textit{in probability},$$

if and only if,

$$\lim_{n\to\infty} \left| \sum_{k=1}^{k_n} a_{nk} X_k(t) \right| = 0, \qquad \textit{in probability}$$

for each $t \in [0, 1]$.

Proof. The "only if" part is trivial. Let $\varepsilon > 0$ be given and choose by (mt) a partition π_m such that

$$\max_{i=1,\ldots,m} E\left[\sup_{t\in I_i} \left| X_n(t) - X_n(t_{i-1}) \right| \right] < \frac{\varepsilon}{4C}, \qquad \forall\, n \in N. \tag{2.1}$$

Write

$$\begin{aligned} P\left[\left\| \sum_{k=1}^{k_n} a_{nk} X_k \right\| > \varepsilon\right] &\leqslant P\left[\max_i \sup_{t\in I_i} \left| \sum_{k=1}^{k_n} a_{nk}(X_k(t) - X_k(t_{i-1})) \right| > \frac{\varepsilon}{2}\right] \\ &\quad + P\left[\max_i \left| \sum_{k=1}^{k_n} a_{nk} X_k(t_{i-1}) \right| > \frac{\varepsilon}{2}\right]. \end{aligned}$$

The second term on the right tends to zero by hypothesis. For the first term on the right, put $Y_k^i = \sup_{t\in I_i} |X_k(t) - X_k(t_{i-1})|$, $i = 1,..., m$, $k \in N$. Then,

$$P\left[\max_{i=1,\ldots,m}\sup_{t\in I_i}\left|\sum_{k=1}^{k_n} a_{nk}(X_k(t)-X_k(t_{i-1}))\right|>\frac{\varepsilon}{2}\right]$$
$$\leqslant P\left[\max_i \sum_{k=1}^{k_n} |a_{nk}|\, Y_k^i > \frac{\varepsilon}{2}\right]$$
$$\leqslant \sum_{i=1}^{m} P\left[\sum_{k=1}^{k_n} |a_{nk}|(Y_k^i - EY_k^i) > \frac{\varepsilon}{2} - \sum_{k=1}^{k_n} |a_{nk}|\, EY_k^i\right]$$
$$< \sum_{i=1}^{m} P\left[\sum_{k=1}^{k_n} |a_{nk}|(Y_k^i - EY_k^i) > \frac{\varepsilon}{4}\right] \qquad \text{(using (2.1))}$$
$$\leqslant \frac{16}{\varepsilon^2}\sum_{i=1}^{m} E\left[\left(\sum_{k=1}^{k_n} |a_{nk}|\,(Y_k^i - EY_k^i)\right)^2\right]$$
$$= \frac{16}{\varepsilon^2}\sum_{i=1}^{m}\sum_{k=1}^{k_n} a_{nk}^2\, E[(Y_k^i - EY_k^i)^2] \qquad \text{(using } (uc))$$
$$\leqslant \frac{16}{\varepsilon^2} m 4 \sum_{k=1}^{k_n} a_{nk}^2\, E(\|X_k\|^2) \to 0, \qquad \text{as} \quad n\to\infty,$$

by hypothesis. Thus

$$\lim_{n\to\infty} P\left[\left\|\sum_{k=1}^{k_n} a_{nk}X_k\right\| > \varepsilon\right] = 0, \qquad \text{for any } \varepsilon > 0. \quad \blacksquare$$

In traditional Banach space results, tightness and moment conditions are used to obtain condition (T) which played a major role in the proofs. However, if (T) is satisfied then (mt) is equivalent to the condition:

If to every $\varepsilon > 0$ and $\delta > 0$ there is a partition π_m such that

$$\|\pi_m\| < \delta \qquad \text{and} \qquad \max_{i=1,\ldots,m} E|X_n(t_i - 0) - X_n(t_{i-1})| < \varepsilon, \text{ for all } n.$$

In this case a weak law holds by using the less restrictive uncorrelation condition (AU) rather than (uc). The proof of Theorem 2.2 follows the same pattern as the proof of Theorem 2.1.

THEOREM 2.2. *Let (X_n) be a sequence of random elements in D satisfying (mt), (T), and (AU). Let $\{a_{nk}\}$, $k = 1,\ldots, k_n$; $n \in N$, be an array of real numbers satisfying $\sum_{k=1}^{k_n} |a_{nk}| \leqslant C < \infty$, for each n, and $\lim_{n\to\infty}\sum_{k=1}^{k_n} a_{nk}^2 E(\|X_k\|^2) = 0$. Then*

$$\lim_{n\to\infty}\left\|\sum_{k=1}^{k_n} a_{nk}X_k\right\| = 0, \qquad \textit{in probability},$$

if and only if

$$\lim_{n\to\infty} \left| \sum_{k=1}^{k_n} a_{nk} X_k(t) \right| = 0, \qquad \textit{in probability}$$

for each $t \in [0, 1]$.

By assuming the much more restrictive condition (MT) instead of (mt), the uncorrelation condition is not needed. The proof of the following weak law is similar to the proof of Theorem 2.1.

THEOREM 2.3. *Let* (X_n) *be a sequence of random elements in* D *satisfying* (MT). *Let* $\{a_{nk}\}$, $k = 1,\ldots, k_n$; $n \in N$, *be an array of real numbers satisfying* $\sum_{k=1}^{k_n} |a_{nk}| \leqslant C < \infty$, *for every* n. *Then*

$$\lim_{n\to\infty} \left\| \sum_{k=1}^{k_n} a_{nk} X_k \right\| = 0, \qquad \textit{in probability,}$$

if and only if

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} a_{nk} X_k(t) = 0, \qquad \textit{in probability}$$

for each $t \in [0, 1]$.

The last result is very similar in appearance to the classical weak law of large numbers for real-valued random variables and uses the uncorrelation condition (C).

THEOREM 2.4. *Let* (X_n) *be a sequence of mean-zero pointwise uncorrelated random elements in* D *satisfying* (C). *Let* $\{a_{nk}\}$, $k = 1,\ldots, k_n$; $n \in N$, *be an array of real numbers satisfying* $\sum_{k=1}^{k_n} |a_{nk}| \leqslant C < \infty$, *for all* $n \in N$. *Suppose that* $\lim_{n\to\infty} \sum_{k=1}^{k_n} a_{nk}^2 E(\|X_k\|^2) = 0$. *Then*

$$\lim_{n\to\infty} \left\| \sum_{k=1}^{k_n} a_{nk} X_k \right\| = 0, \qquad \textit{in probability.}$$

Proof. Let $\varepsilon > 0$ be given and choose by (C) a partition π_m such that

$$\sup_{k\neq l} E\left[\max_{i=1,\ldots,m} \sup_{t\in I_i} |X_k(t) X_l(t) - X_k(t_{i-1}) X_l(t_{i-1})| \right] \leqslant \frac{\varepsilon^3}{3C^2}. \tag{2.2}$$

Let n_0 be such that

$$\sum_{k=1}^{k_n} a_{nk}^2 E(\|X_k\|^2) < \frac{\varepsilon^3}{6m}, \qquad \forall\, n \geqslant n_0. \tag{2.3}$$

We have

$$
\begin{aligned}
P&\left[\left\|\sum_{k=1}^{k_n} a_{nk} X_k\right\| \geqslant \varepsilon\right] \\
&\leqslant \frac{1}{\varepsilon^2} E\left(\left\|\sum_{k=1}^{k_n} a_{nk} X_k\right\|\right)^2 = \frac{1}{\varepsilon^2} E\left[\sup_{0\leqslant t\leqslant 1}\left|\sum_{k=1}^{k_n} a_{nk} X_k(t)\right|^2\right] \\
&= \frac{1}{\varepsilon^2} E\left[\max_i \left\{\sup_{t\in I_i} \sum_{k,l=1}^{k_n} a_{nk} a_{nl} (X_k(t) X_l(t) - X_k(t_{i-1}) X_l(t_{i-1}))\right.\right. \\
&\quad \left.\left. + \sum_{k,l=1}^{k_n} a_{nk} a_{nl} X_k(t_{i-1}) X_l(t_{i-1})\right\}\right] \\
&\leqslant \frac{1}{\varepsilon^2} E\left[\max_i \sup_{t\in I_i} \sum_{k,l=1}^{k_n} |a_{nk}|\,|a_{nl}|\,|X_k(t) X_l(t) - X_k(t_{i-1}) X_l(t_{i-1})|\right] \\
&\quad + \frac{1}{\varepsilon^2} E\left[\max_i \sum_{k,l=1}^{k_n} a_{nk} a_{nl} X_k(t_{i-1}) X_l(t_{i-1})\right] = (\mathrm{I}) + (\mathrm{II}).
\end{aligned}
$$

For $n \geqslant n_0$,

$$
\begin{aligned}
(\mathrm{I}) &= \frac{1}{\varepsilon^2} E\left[\max_i \sup_{t\in I_i} \sum_{k,l=1}^{k_n} |a_{nk}|\,|a_{nl}|\,|X_k(t) X_l(t) - X_k(t_{i-1}) X_l(t_{i-1})|\right] \\
&\leqslant \frac{1}{\varepsilon^2} E\left[\max_i \sup_{t\in I_i} \sum_{\substack{k,l=1\\ k\neq l}}^{k_n} |a_{nk}|\,|a_{nl}|\,|X_k(t) X_l(t) - X_k(t_{i-1}) X_l(t_{i-1})|\right] \\
&\quad + \frac{1}{\varepsilon^2} E\left[\max_i \sup_{t\in I_i} \sum_{k=1}^{k_n} a_{nk}^2 |X_k^2(t) - X_k^2(t_{i-1})|\right] \\
&\leqslant \frac{C^2}{\varepsilon^2} \frac{\varepsilon^3}{3C^2} + \frac{2}{\varepsilon^2} \sum_{k=1}^{k_n} a_{nk}^2 E(\|X_k\|^2) \qquad (\text{by } (2.2)) \\
&< \frac{\varepsilon}{3} + \frac{2}{\varepsilon^2} \frac{\varepsilon^3}{6m} \leqslant \frac{2}{3}\varepsilon \qquad (\text{by } (2.3)).
\end{aligned}
$$

By (2.3) and pointwise uncorrelation,

$$
\begin{aligned}
(\mathrm{II}) &\leqslant \frac{1}{\varepsilon^2} \sum_{i=1}^{m} E\left[\sum_{k,l=1}^{k_n} a_{nk} a_{nl} X_k(t_{i-1}) X_l(t_{i-1})\right] \\
&= \frac{1}{\varepsilon^2} \sum_{i=1}^{m} \sum_{k=1}^{k_n} a_{nk}^2 E[X_k^2(t_{i-1})] \\
&\leqslant \frac{m}{\varepsilon^2} \sum_{k=1}^{k_n} a_{nk}^2 E(\|X_k\|^2) < \frac{m}{\varepsilon^2} \frac{\varepsilon^3}{6m} < \frac{\varepsilon}{3},
\end{aligned}
$$

whenever $n \geqslant n_0$. Thus,

$$P\left[\left\|\sum_{k=1}^{k_n} a_{nk}X_k\right\| \geqslant \varepsilon\right] \leqslant (\mathrm{I}) + (\mathrm{II}) < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

for $n \geqslant n_0$. Since ε is arbitrary,

$$\lim_{n\to\infty} \left\|\sum_{k=1}^{k_n} a_{nk}X_k\right\| = 0, \qquad \text{in probability.} \quad \blacksquare$$

3. Comparisons and Implications

In considering the different uncorrelation conditions in the hypotheses of the results, it is easy to see that (MT) implies (mt). However, it is not as easy in many of the other comparisons. Since every separable Banach space can be embedded isometrically in the universal Banach space $C[0, 1]$ (which can be regarded as a subspace of D), previous results for separable Banach spaces can be compared somewhat with these results in the more general setting of D. First, the conditions for the weak law of large numbers in non-geometric Banach spaces almost always include tightness and integral conditions (or (T)) and weak uncorrelation. But (T) implies uniform integrability, in which case convex tightness (which is equivalent to tightness in a Banach space) is equivalent to the restrictive condition (MT). Thus, many previous Banach space results are contained in Theorem 2.4 as corollaries.

Weak uncorrelation is defined using the dual space of a Banach space and implies pointwise uncorrelation in any subspace of D which is a Banach space. However, the less restrictive pointwise uncorrelation can be defined on all of D. Moreover, pointwise uncorrelation can yield the pointwise convergence in Theorems 2.1, 2.2, and 2.3. Also, it is not hard to show that if (T) is satisfied, then (MT) is equivalent to the following weaker condition:

$(*)$ To every $\varepsilon > 0$ and every $\delta > 0$, there is a partition π_m such that $\|\pi_m\| < \delta$ and

$$\sup_{n\in N} E\left[\max_i |X_n(t_i - 0) - X_n(t_{i-1})|\right] \leqslant \varepsilon.$$

That $(*)$ alone, in Theorems 2.1, 2.2, and 2.4, would not suffice to turn pointwise convergence in probability into uniform convergence, is shown by the following example.

Example 3.1. Construct the sequence $x_1, x_2, \ldots,$ of elements of D as follows: Let $x_1 = I_{[1/5, 2/5)} - I_{[3/5, 4/5)}$. Supposing x_n is given, construct x_{n+1}

as follows. Where $x_n = 0$, let $x_{n+1} = 0$. If $x_n = 1$ or -1 on an interval $J \subset [0, 1]$, then divide J into five equal subintervals; on the second subinterval of J put $x_{n+1} = 1$, on the fourth subinterval of J, put $x_{n+1} = -1$. Elsewhere on J put $x_{n+1} = 0$. This defines x_{n+1}. Thus, for example,

$$x_2 = I_{[6/25,7/25)} - I_{[8/25,9/25)} + I_{[16/25,17/25)} - I_{[18/25,19/25)}.$$

Now define mean-zero random elements (X_n) in D by: $X_n = \pm x_n$, with probability $\frac{1}{2}$ each.

For this sequence (X_n), $(*)$ is satisfied. Indeed, let $\delta > 0$ be given and choose m such that $5^{-m} < \delta$. Let π_m consist of the points $t_i = i \cdot 5^{-m}$, $i = 0, 1, \ldots, 5^m$. If $n \geqslant m$, then $X_n(t_i) = 0$, $\forall\, i = 0, 1, \ldots, 5^m$. If $n < m$, then $|X_n(t_i - 0) - X_n(t_{i-1})| = 0$, $i = 1, \ldots, m$ (replacing $t_m - 0$ by $t_m = 1$), since an interval $J \subset [0, 1]$ on which X_n is constant is a union of intervals of the form $[i \cdot 5^{-m}, (i+1)\, 5^{-m})$. Thus, $\max_{i=1,\ldots,m} |X_n(t_i - 0) - X_n(t_{i-1})| \equiv 0$, for all n so that $(*)$ holds.

That $\|n^{-1} \sum_{k=1}^{n} X_k\| = 1$ a.s., for every n can be seen inductively as follows: Clearly $\|X_1\| = 1$ a.s. Let $S_n = \sum_{k=1}^{n} X_k$. If for almost every realization, $X_1(t) = \cdots = X_n(t) = 1$ for each t in an interval J of the form $[i \cdot 5^{-m}, (i+1)\, 5^{-m})$, then by the way x_{n+1} was constructed from x_n, $X_{n+1}(t) = 1$ for some $t \in J$. Thus, $S_{n+1}(t) = n+1$, and $(n+1)^{-1} \sum_{k=1}^{n+1} X_k(t) = 1$, for this t. Since the induction hypothesis is satisfied a.s. for $n = 1$, we have $\|n^{-1} \sum_{k=1}^{n} X_k\| = 1$ a.s., for every n as claimed. Thus, the strong law of large numbers fails for this sequence (X_n), and the weak law of large numbers must also fail.

Conditions (T) and (mt) are natural extensions of the conditions on identical distributions and Lemma 8.1 of [4] which are used in obtaining ergodic theorems for $D[0, 1]$. Condition (AU) appears to be very similar to the concept of weak uncorrelation which was used in [5] to obtain WLLNs for Banach spaces. But more importantly, (uc), (C), and (AU) are easily applicable to stochastic processes which can be regarded as random elements in D. Conditions on increments of stochastic processes are very natural considerations.

In comparing the uncorrelation concepts, it is sufficient to consider only (uc), (C), and pointwise uncorrelation. If A and B are two correlated real-valued random variables, then $X(t) \equiv A$ and $Y(t) \equiv B$ for all $t \in [0, 1]$ are random elements in D which are not pointwise uncorrelated but do satisfy condition (C). Next, $X_n(t) = (1 - tn)\, I_{[0,1/n)}(t)$ defines a deterministic sequence of random elements in Banach space $C[0, 1]$ and as such is pointwise uncorrelated. But if $n \neq k$ and $n, k > t_1^{-1}$, then

$$\sup_{0 \leqslant t \leqslant t_1} |X_n(t)\, X_k(t) - X_n(0)\, X_k(0)| = 1.$$

Hence, condition (C) does not hold for the sequence (X_n). Note also, that condition (uc) holds for this sequence $\{X_n\}$. These examples illustrate that uncorrelation in $D[0, 1]$ is related to both the global probability structure and variation in the sample paths. Similar examples show that no implications exist between any of the uncorrelation conditions (uc), (C), and weak uncorrelation.

References

[1] Billingsley, P. (1968). *Convergence of Probability Measures*. Wiley, New York.

[2] Daffer, P. Z. (1981). Some strong and weak laws of large numbers for weighted sums in $D[0, 1]$. In *Probability in Banach Spaces III* (Proceedings, Medford, 1980), pp. 99–106. Lecture Notes in Mathematics No. 860, Springer-Verlag, Berlin/Heidelberg/New York.

[3] Daffer, P. Z., and Taylor, R. L. (1979). Laws of large numbers for $D[0, 1]$. *Ann. Probab.* **7** 85–95.

[4] Parthasarathy, K. R. (1967). *Probability Measures on Metric Spaces*. Academic Press, New York/London.

[5] Taylor, R. L. (1978). *Stochastic Convergence of Weighted Sums of Random Elements in Linear Spaces*. Lecture Notes in Mathematics No. 672, Springer-Verlag, Berlin/Heidelberg/New York.

[6] Taylor, R. L., and Daffer, P. Z. (1980). Convergence of weighted sums of random elements in $D[0, 1]$. *J. Multivariate Anal.* **10** 95–106.